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Groupoids: a local theory of symmetry

Angelo Vistoli

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

Email: angelo.vistoli@sns.it

1. Introduction

The theme of symmetry is of great interest to mathematician, physicists, chemists, biologists, psychologists, philosophers, and others. The very word "symmetry" is used with a wide variety of meanings; I will only discuss the way it is used in mathematics.

In fact, even this seems to me too ambitious a goal. Symmetry permeates every field of mathematics, and I do not have the intention, and even less the ability, to give a comprehensive picture of its multifaceted aspects.

The mathematical analysis of the concept has been traditionally based on the theory of group actions. As we shall discuss, this notion is global; that is, the symmetries of a structure (geometric or otherwise) always involve the whole structure. It is natural, on the other hand, to talk about *local symmetries*, symmetries that appear only among certain parts of the structure itself. Mathematicians have a local theory of symmetry, which is known as the theory of groupoids.

However, its existence does not seem to have been really noticed outside of the communities of mathematicians and theoretical physicists; the only place in the philosophical literature where I have seen it discussed is Corfield's book (Corfield 2003), which, I am afraid, has not been read by many philosophers, because of the vast mathematical background it requires.

The very modest purpose of this note is to give a quick introduction to symmetry in mathematics, and point out the existence of a mathematical analysis of the notion of local symmetry to philosophers and others who may be interested in this theme. No originality whatsoever is claimed for any of the ideas presented here.

2. The classical point of view: symmetry via group actions

In this section we give a quick introduction to the classical point of view on symmetry in mathematics.

Let us venture to give a general, necessarily vague, definition of symmetry in mathematics. A *symmetry* is a transformation that preserves certain properties. Two objects are *symmetric* if they can be obtained from each other with such a transformation. Hence, by definition, two symmetric objects have some common properties, the one preserved by the transformation.

Historically, one of the first examples comes from euclidean geometry. One of the basic notions is that of *congruence* (formerly known as *equality*; but today the latter word is better reserved for the relation of identity). Two geometric figure are congruent when they can be superimposed. What does this mean? In Euclid's *Elements* this notion, while constantly used, is never defined. Here is the modern definition.

An *isometry* is a transformation of the euclidean plane $\mathbb{E}^2$ onto itself which preserves distances. Here is a more precise definition. We consider $\mathbb{E}^2$ as a collection of points, that is, as a set whose elements are points[1]. Given two points p and q in $\mathbb{E}^2$, there is a non-negative real number $\mathrm{d}(p,q)$ called the *distance* of p and q.[2] An isometry of the euclidean plane is a bijective transformation $f\colon \mathbb{E}^2 \to \mathbb{E}^2$ such that for any two points p and q of $\mathbb{E}^2$, the distance between pf and qf (here we indicate with pf the transform of p along f, instead of the more traditional $f(p)$) equals the distance between p and q.[3]

Two figures T and T' in $\mathbb{E}^2$ are *congruent* if there exists an isometry of $\mathbb{E}^2$ that carries T onto T'. It turns out that all the other concepts of euclidean geometry (angles, areas, ...) can be expressed in terms of distance; hence two congruent figures have the same properties in euclidean geometry.

This definition of isometry is quite general, and one might suspect that it includes wild operations never considered by Euclid; but this is not the case, as all isometries are either translations, rotations, reflexions along a line or glide-reflexions (a *glide-reflexion* is the composite of a reflexion along a line with a translation along a vector parallel to the line).

It is a very important fact that isometries can be composed. If f and g are two isometries, we indicate by fg the transformation that we obtain by first applying f, then g (in symbolic terms, if p is a point of $\mathbb{E}^2$ we have $p(fg) = (pf)g.$). The composite fg is easily checked to be again an isometry. Also, because an isometry f is bijective, we have the inverse transformation $f^{-1}\colon \mathbb{E}^2 \to \mathbb{E}^2$, which can be seen to be again an isometry.

We will denote by Isom_2 the set of all isometries in the plane. Then Isom_2 is a group of transformations of $\mathbb{E}^2$, in the following sense. Let M be a set. A *group of transformations* of M is a set G of bijective transformations sending points of M in points of M, in such a way that:

(a) The identity id_M[4] is in G.

[1]Of course, this leave open the question of what $\mathbb{E}^2$ really is. One possibility is to identify $\mathbb{E}^2$ with the cartesian plane $\mathbb{R}^2$; while this works quite very in practice, it is conceptually somewhat unsatisfactory, as it endows $\mathbb{E}^2$ with a preferred element, the origin $(0,0)$, and two preferred directions, corresponding to the two coordinate axis. A better solution is not to give a univocal definition of $\mathbb{E}^2$, but say that *a* euclidean plane is a principal homogenous space under a two-dimensional real vector space with a positive defined inner product. (Of course will probably fly right over the head of most readers.) Or one can give an axiomatic treatment of $\mathbb{E}^2$, as Hilbert did (see (Hilbert 1899) for an English translation), without making a reference to the real numbers; this is not, however, the point of view that we want to take here.

[2]If $\mathbb{E}^2$ is identified with $\mathbb{R}^2$, then the points p and q are pairs (p_1, p_2) and (q_1, q_2) of integers, and the distance $\mathrm{d}(p,q)$ is given by the usual expression $\sqrt{(p_1-q_1)^2 + (p_2-q_2)^2}$, coming from Pythagoras' Theorem.

[3]It can be shown that the condition that f be bijective is actually automatically satisfied, so it does not need to be part of the definition, but this is non-trivial.

[4]This is the transformation sending each point of M into itself.

(b) If f and g are in G, their composite[5] fg is also in G.
(c) If f is in G, then the inverse transformation f^{-1} is also in f.

We obtain an example by taking $M = \mathbb{E}^2$ and $G = \mathrm{Isom}_2$.

Here is a very general construction. Every time we have some kind of "space" (considered, usually, as a set of points), or more generally, a "structure" M, we can consider the set of transformations of M into itself that preserve the structure under consideration (in the case of the euclidean plane $\mathbb{E}^2$, the structure was given by the distance). These transformations are called *automorphisms* of M, and form a group of transformations, often denoted by $\mathrm{Aut}\, M$. Let us consider some examples, again coming from geometry.

Another form of geometry that has a very old history is *spherical geometry*, that is, the geometry of the surface of a sphere. It was first developed (of course, as a part of euclidean geometry) by the astronomer Menelaus of Alexandria in the first century B.C.; he was the first to recognize that great circles[6] played a role very analogous to that of lines in plane geometry. This geometry has an automorphism group, called the orthogonal group in three dimensions O_3; its elements consist of rotations along an axis passing through the center of the sphere, reflexions along planes through the center, and their composites.

Another very important form of geometry that emerged in the 17th century is *projective geometry*. Inspired by the studies of perspective of Renaissance painters such as Brunelleschi, Kepler and, above all, Desargues, followed by Poncelet and von Staudt at the beginning of the 19th century, have introduced a form of plane geometry in which distance is no longer defined, and the central concept is that of incidence between line and point. It is truly remarkable that one do something interesting with so little. The transformations that conserve projective properties are those sending lines into lines; they form a group, the real projective group in two dimensions $\mathrm{PGL}_2(\mathbb{R})$.

In the first half of the 19th century Gauss, Lobachevsky and Bolyai introduced hyperbolic geometry. A fundamental step for our history was taken by Eugenio Beltrami, Arthur Cayley and Felix Klein, who gave a metric model for hyperbolic geometry. In this way one can discuss automorphisms of this geometry; these form a group, the real special projective group in two dimensions $\mathrm{PSL}_2(\mathbb{R})$.

In all these cases one has a space M with some structure, and a group of transformations G of M preserving the structure.

In 1872 Klein gave to light his "Erlangen program" (see for example (Klein 1924, Part 3)). In technical terms, this consists of defining geometry as the study of homogeneous spaces. "Geometry" is defined as the study of those properties that are invariant under a group of transformations. For example, euclidean geometry is defined as the study of the properties of "figures" that are invariant under isometries. Spherical and hyperbolic geometry can be defined similarly.

A group of transformations G on a set M is called *homogeneous* if every point of M can be carried to any other point via an element of G. According to Klein, "geometry" is the study of properties of subsets of M that are invariant by the transformations in G.

[5] We define the composite fg by first applying f and then g, as in the case of isometries. This is against the usual convention, but has some advantages.
[6] That is, the circles cut on a sphere by a plane passing through its center.

It is a great idea, but like all ideas it has some limits. In fact, in some sense it was outdated already at birth: in 1868 Riemann published his fundamental work "Über die Hypothesen welche der Geometrie zu Grunde liegen" (for an English translation, see (Riemann 1854)), in which the idea of metric geometry, that is, geometry based on a notion of distance, is expanded far beyond the realm of homogenous spaces.

This said, the geometry of homogeneous spaces is extremely important even today.

From the notion of transformation group there slowly emerged in the 19^{th} century, formalized in the 20^{th} century a notion of *abstract group*, now simply called a *group*. A group is a set G, with an operation that associates with every pair of elements g and h of G another element of G, usually denoted by gh, in such a way that

(1) there exists an element 1 of G, called *the identity*, such that $1g = g1 = g$ every g in G;
(2) the operation is associative, that is $(gh)k = g(hk)$ for any g, h an k in G, and
(3) every element g has in inverse g^{-1}, with the property $gg^{-1} = g^{-1}g = 1$.

Starting from these axioms, one shows that the identity and the inverse of an element are unique.

Groups appear almost every in mathematics; despite the apparent simplicity of the axioms, the theory of groups is enormously complex.[7]

Most groups arise as groups of transformations, but not all of them.

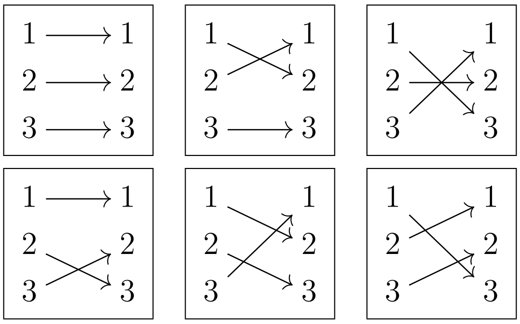
One should not be left with the impression that symmetry is important only in geometry; in fact, it plays a role in all the fields of mathematics (at least, I am not aware of any exception). Let us see an example, *Galois groups*, which are very important in number theory.

Take a finite set $X = \{1, \ldots, n\}$, without a particular structure. The set of all symmetries is denoted by S_n, and is called *the symmetric group on n symbols*. It has $n! = 1 \times 2 \times (n-1) \times n$ elements. For $n = 2$ it has two elements:

$$\boxed{\begin{tikzcd} 1 \arrow[r] & 1 \\ 2 \arrow[r] & 2 \end{tikzcd}} \qquad \boxed{\begin{tikzcd} 1 \arrow[dr] & 1 \\ 2 \arrow[ur] & 2 \end{tikzcd}}$$

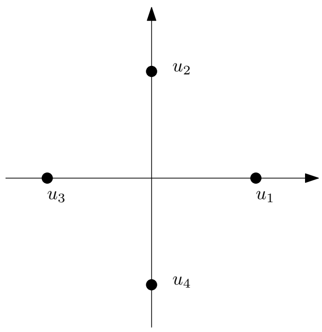
For $n = 3$ it has 6:

[7]This system of axioms is quite different in spirit from Euclid's and Hilbert's axiom systems for euclidean geometry. The latter are foundational in nature, and have the aim of pinning down as precisely as possible a unique model that one has in mind. The system of axioms for groups, instead, identifies a wide class of structures. Cultured laymen, for example philosophers who have done some reading in the history of mathematics, may be left with the impression that "axiomatics" in mathematics is important because mathematicians care about rigor and foundations, and want their concepts defined with as much precision as possible, through systems of axioms like Hilbert's (for examples, the axioms for set theory). In fact, systems of axioms are used pervasively in mathematics for non-foundational purposes, but they are almost always of the first kind, that is, they are meant to single out a whole class of structures, which can then interact among themselves, and with other kinds of structures, instead of a single structure in which all the action takes place.



The operation in this group is given by composition.

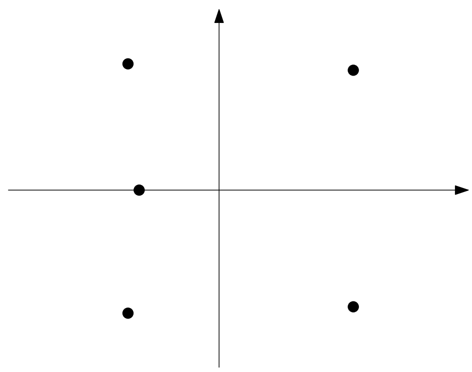
Take a polynomial of degree n, $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$, with rational coefficients. It has n roots $u_1, \ldots, u_n$ in the complex plane $\mathbb{C}$. These are subjects to various algebraic relations with rational coefficients. As an example, take the polynomial $x^4 - 2$. Its complex roots are $u_1 = \sqrt[4]{2}$, $u_2 = i\sqrt[4]{2}$, $u_3 = -\sqrt[4]{2}$, $u_3 = -i\sqrt[4]{2}$.



The roots are subject to various algebraic relations: for example, $u_1 + u_3 = 0$, $u_2 + u_4 = 0$, $u_1^2 + u_2^2 = 0$.

Consider the group S_n of permutations of $u_1, \ldots, u_n$. The *Galois group of $f(x)$* is the group of permutations in S_n which conserve all the relations. In the example, the permutation that swaps u_3 e u_4 keeping u_1 e u_2 fixed is not in the Galois group of $x^4 - 2$, because $u_1 + u_3 = 0$ while $u_1 + u_4 \neq 0$. One shows that the Galois group of $x^4 - 2$ contains 8 elements, represented by the symmetries of the square formed by the roots, as in the picture above; this is called a *dihedral group*, and is denoted by D_4.

However, this example is too simple, as it might induce to believe that the algebraic symmetries of a polynomial, that is, the elements of its Galois group, come from geometric symmetries of the roots. This is completely false. For example, the roots of $x^5 + 4x + 3$ are pictured below



The only symmetry that is evident from the diagram is a reflexion along the real axis, while in fact the Galois group of $x^5 + 4x + 3$ is the whole symmetric group S_5; that is, any permutation of the roots gives an algebraic symmetry.

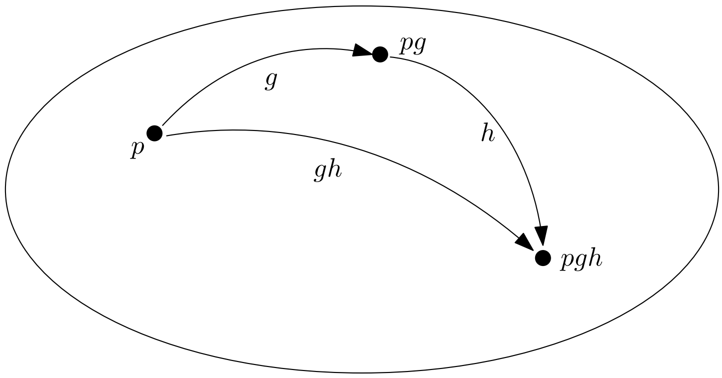
But sometimes in the study of an object X we are not interested, or can't control, all the symmetries, but only some of them. Furthermore, an absolutely fundamental technique to study a group is to let it *act* as a symmetry group on some other structure. For example, an enormous branch of mathematics, with manifold aspects, called *representation theory*, studies precisely how a group can act via linear transformations.

A *action* of a group G on a structure X consists in the assignment of a symmetry of X to each element of G, in such a way that if g and h are in G, the symmetry assigned to gh is the composite of the transformation assigned to g and that assigned to h. If an action of G on X is given, we say that G *acts on* X.

An action of G on X is often given as a function $X \times G \to X$, denoted simply with $(p, g) \mapsto pg$. It must satisfy the following conditions.

(1) $p1 = p$.
(2) $p(gh) = (pg)h$.

This can be visualized as follows.



An action of G on X is *transitive* if, given any two elements p and q of X, there exists g in G that carries p onto q (that is, such that $gp = q$).

More generally, given an action of a group G on a space X, and given a point p of X, one defines the *orbit* of p as the set of points of X that can be reached from p. In other words, the orbit of p is the set of points pg, where g is an element of the group.

To give an example, if $\mathbb{E}^2$ is the euclidean plane, its groups of symmetries Isom_2 acts on it. But there is also an action of Isom_2 on the cartesian product $\mathbb{E}^2 \times \mathbb{E}^2$, given by $(p, q)g = (pg, qg)$. Given (p, p') and (q, q') in $\mathbb{E}^2 \times \mathbb{E}^2$, when is (q, q') in the orbit of (p, p')? The answer is: if and only if the distance between q and q' equals the distance between p and p'.

This point of view is enormously important in mathematics. To study a structure we are interested in its group of symmetries (or automorphisms). But to study the group of automorphisms, it can be extremely useful to make it act on other structure.

Furthermore, it should be pointed out that groups often has an additional geometric structure. We call them *discrete groups* when we are only interested in their algebraic structure; often, however, they have a natural *topology*, or a more refined geometric structure. Then we talk about Lie groups, topological groups, algebraic groups, and so on[8].

For instance, take the groups Isom_2 of plane isometries: we can talk about two isometries being "close" or "distant". For example, two translations of the plane along two vectors whose difference has very small length should be considered as close. As a more sophisticated example, let us imagine to take rotations of a smaller and small angle, whose center of rotation move further and further away. If this is done appropriately, these rotation get closer and closer to a translation. On the other hand, one cannot get closer and closer to a reflexion with a sequence of rotations; rotation and reflexions are always "distant". In technical terms, they belong to different connected components of Isom_2. The group Isom_2 has a natural structure of a Lie group.

The theory of groups, both discrete and non-discrete, pervades most of mathematics. Let me only mention a very important result, and an open question.

The problem of the classification of all finite groups (describing *all* finite groups) is exceedingly complex, completely inapproachable at the present time. However, in the first half of 1980's all the finite simple groups have been classified. Finite simple groups are, in some sense, the bricks with which all finite groups are built. It is one of the most complex mathematical results ever achieved: finite simple groups form 18 different infinite families, with 26 exceptional groups, not included in any of the families. The largest of the exceptional groups is known as *the monster*[9]; it has 808.017.424.794.512.875.886.459.904.961.710.757.005.754.368.000.000.000 elements.

The following problem is still open. What are the finite groups that can appear as Galois groups of polynomials with rational coefficients? It is conjectured that all of them can, but this has not been proved.

Thus, the following slogan could be, and has been, adopted.

The mathematical theory of symmetry is the study of group actions.

3. The local point of view: symmetry via groupoids

Like most slogans, this contains some truth, but also some falsehood[10].

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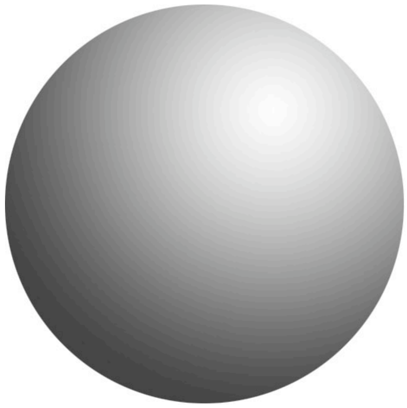
[8]Thus, Lie groups, and so on, are not particular kinds of groups, different from discrete groups, but groups with additional structure. One can take a Lie group, forget the additional structure, retaining only the algebraic structure, and get a discrete group.
We should also notice, however, that the discrete groups that one obtains this way are not very interesting; the discrete groups that are amenable to being analyzed with our tools are those satisfying some kind of finiteness condition, while the discrete groups that one obtains from Lie groups are much too large. In other word, when working with Lie groups, the geometric structure interacts in an essential way with the algebraic structure, and the interest and richness of the theory arises from this interaction.

[9]Amusingly, there have been attempts to rename it *the friendly giant*, but they did not catch.

[10]I say "most slogans" because some slogans only contain falsehood.

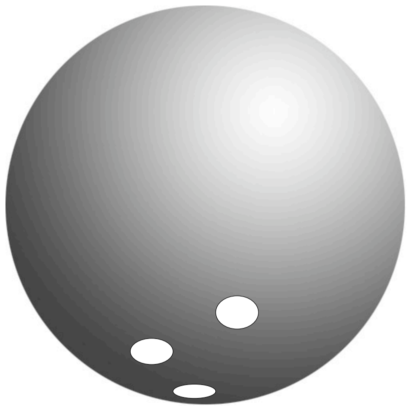
As a theory of symmetry, the theory of group actions is not completely satisfactory. Its main problem is that it is a *global* theory. The symmetries of a structure, as we have defined them, always involve the whole structure.

Let us imagine a spherical surface, depicted below[11].



This has a large symmetry group, composed of the rotations along an axe passing through the center of the sphere[12]. These form a group, denoted by SO_3.

Now, let us imagine drilling three holes in the sphere in irregular position.



After this act of vandalism, our sphere has only one symmetry, namely the identity. But is it really reasonable to say that the sphere with holes has no non-trivial symmetry? Choose two points p and q of the pierced sphere. Unless p or q is very near one of the holes, the sphere looks very much the same close to p and close to q, so it seems like there should be a symmetry carrying p onto q. In other words, shouldn't we be able to say that the sphere has symmetries in certain areas, but not everywhere?

So we are asking for a local theory of symmetry. Well, such a mathematical theory exists. It is the theory of groupoids[13].

[11]We will call this spherical surface simply *a sphere*. In mathematics by a sphere one always intends a spherical surface, and not a solid sphere, which is called a *ball*.

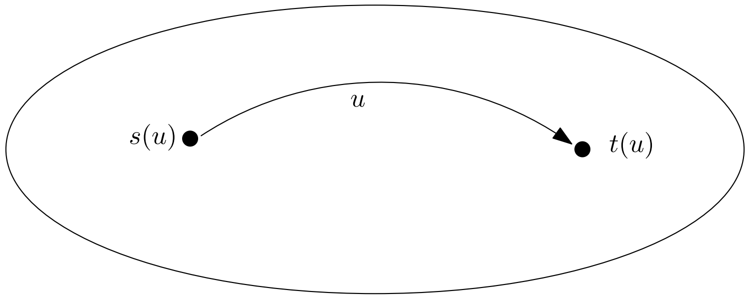
[12]We are only considering the *orientation preserving* symmetries; there are other, such as the reflexions along a plane passing through the center, which reverse the orientation of the sphere.

[13]Of course, asking for "a theory of symmetry" is not something that a mathematician would do, it is more appropriate for a philosopher. Like all scientists, mathematician tend to be pragmatic, and build a theory only because it is needed in order to study some specific phenomena. For example, the definition of groupoid (in a slightly different form from the one I will give below) first appeared

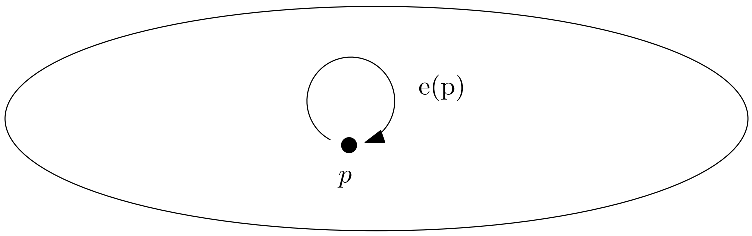
What is a groupoid[14]? Let us start from an example.

Consider the spheric surface $\mathbb{S}^2$ as above, and let SO_3 be the group of rotations along an ass passing through the center of $\mathbb{S}^2$. Then SO_3 acts on $\mathbb{S}^2$; an element g of SO_3 can be applied to a point p of $\mathbb{S}^2$, obtaining another point pg of $\mathbb{S}^2$. Now, call X our surface with holes; we can think of X as a subset of $\mathbb{S}^2$, the set of points of $\mathbb{S}^2$ that are not in one of the three holes. The group SO_3 does not act on X: if g is in SO_3 and p is in X, the point pg of $\mathbb{S}^2$ could be in one of the holes, so not in X. The solution consists of considering the set R of pairs (p, g) in the cartesian product $X \times G$, with the property that pg is in X. We can think of such a pair (p, g) as a symmetry from p to pg. Thus, an element g of SO_3 does not give a global symmetry, because pg is not always in X, but it gives a symmetry from certain parts of X to other parts.

This set of symmetries R has a rather complicated structure. First of all, there are functions $s\colon R \to X$ and $t\colon R \to X$; the first sends (p, g) into $s(p, g) = p$, the second into $t(p, g) = pg$. So, for each u in R, we think of u as a symmetry from $s(u)$ to $t(u)$[15], as in the following picture.



Furthermore, if p is in X, then the pair $(p, 1)$ is in R. In fact $p1$ is p, which is in X by hypothesis. Hence, there is a function $e\colon X \to R$ sending p into the $(p, 1)$. This element plays the role of an identity; but unlike the theory of groups, there is not just one identity, but there is one for each element of X.



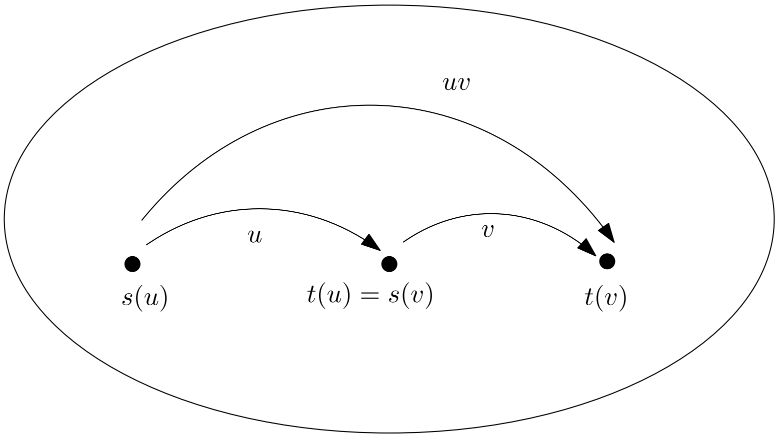
There is also a composition operation. Assume that (p, g) and (q, h) are in R. If $pg = q$, then $p(gh) = (pg)h = qh$ is in X; hence we can define the *composite* $(p, g)(q, h) = (p, gh)$. So, we have an operation of composition $c\colon \widetilde{R} \to R$, where $\widetilde{R}$ is the set of elements (u, v) of $R \times R$ such that $t(u) = s(v)$. As above, we denote the composite of u and v simply by uv. A groupoid will have an operation, but, unlike

in a 1927 article (Brandt 1927) by Heinrich Brandt, who was motivated by a concrete problem in algebra, that of the composition of quadratic forms.

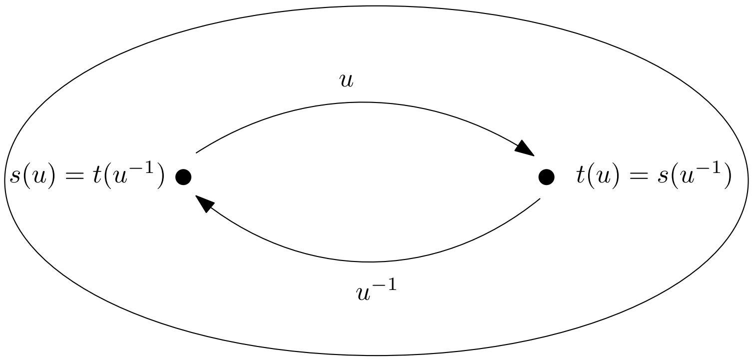
[14]In some older literature a *groupoid* is defined as a set with a binary operation, without any hypothesis whatsoever. This is obsolete terminology; such a structure is nowadays called a *magma*.

[15]The letters s and t come from the English words *source* and *target*.

in the case of groups, this operation will not be defined for all pairs.



Finally, if (p, g) is in R, then $(pg)g^{-1} = p(g^{-1}g) = p1 = p$; the pair (pg, g^{-1}) is called *the inverse* of (p, g), and goes from pg to p. We denote the inverse of an element u by u^{-1}; call $i\colon R \to R$ the function that sends u onto u^{-1}.



These two operations satisfy a collection of obvious identities.

(1) $s(uv) = s(u)$, $t(uv) = t(v)$;
(2) $s\big(e(p)\big) = t\big(e(p)\big) = p$;
(3) $s(u^{-1}) = t(u)$ and $t(u^{-1}) = s(u)$;
(4) $(uv)w = u(vw)$ (associativity);
(5) $e\big(s(u)\big)u = u$ and $ue\big(t(u)\big) = u$ (the $e(p)$ are identities);
(6) $uu^{-1} = e\big(s(u)\big)$ and $u^{-1}u = e\big(t(u)\big)$ (u^{-1} is the inverse of u).

A *groupoid* consists of two sets R and X, with functions $s, t\colon R \to X$, $c\colon \widetilde{R} \to R$ e $i\colon R \to R$ satisfying the conditions above.

Here are some examples.

(1) An action of a group G on a set X gives a groupoid, with $R = X \times G$.
(2) An equivalence relation on a set X gives a groupoid.

Recall that an *relation* on a set X is given by a subset $R \subseteq X \times X$.[16]

An equivalence relation on X is a relation $R \subseteq X \times X$ with the following properties:

(a) it is reflexive: (p, p) is in R for every p;
(b) it is symmetric: if (p, q) is in R, then (q, p) is in R;

[16] Every property of pairs of elements of X gives a relation, the set of pairs (p, q) which have this property. This uncompromisingly extensional use of the word "relation" is very widespread in mathematics.

(c) it is transitive: if (p, q) and (q, r) are in R, then (p, r) is also in R.

From an equivalence relation $R \subseteq X \times X$ we obtain a groupoid: $s\colon R \to X$ are defined by $s(p, q) = p$, $t(p, q) = q$, the composition is defined by $(p, q)(p, r) = (p, r)$, and $(p, q)^{-1} = (q, p)$.

(3) The groupoid of symmetries of a bundle.

The notion of *bundle* is a very important one. A bundle E on a space X consists of a structure of a certain type E_p for any point p of X.[17] The groupoid R of symmetries of E on X is defined as follows: an element of R consists of a triplet (p, q, f), were p and q are points of X, and f an isomorphism of the structure E_p with the structure E_q.

Why are groupoids important in mathematics, and how are they used? Here the discussion should become very technical, and I will have to limit myself to giving some hints, and pointers to the literature for a more thorough introduction.

From the strictly algebraic point of view, that is, without an added geometric structure, the theory does not differ in an essential way from the theory of groups; in particular, for example, the problems of classification of finite groupoids is essentially the same as the problem of classifying finite groups. However, groupoids are particular cases of a more general concept, that of *category* (a category is like a groupoid, without the function $i\colon R \to R$). The theory of categories has become very pervasive in mathematics; it has a very different flavor from algebra, and we can say that passing from groups and groupoids adds a level of complexity to the analysis.

But one starts to really see the power of the theory of groupoids when one considers groupoids R over X when R and X have a geometric structure. For example, the theory of Lie groupoids is a very wide extension of the theory of Lie groups, and has a range of applications that is much broader. It also gives an approach to very deep questions, such as the construction of spaces of orbits.

So, let us formulate a new and improved slogan.

The mathematical theory of symmetry is the study of groupoids.

Suggestions for further readings. A very good introductory paper on the theory of groupoids is (Weinstein 1996). Another useful survey, with an overview of the history of the subject, is (Brown 1987). The bibliographies of these two articles contain many references; however, the rest of the literature I am familiar with is written for mathematicians, and is not easy to approach for a layman. The book (Higgins 2005) gives an introduction to the categorical aspects of the theory of groupoids, and is relatively accessible. For the theory of Lie groupoids, a standard reference is (Mackenzie 2005). For the applications to algebraic topology one can consult (Brown 2006).

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[17] There are many kinds of bundles, according to what kinds of spaces we are considering, and what kind of structure we put on the E_p. I am being very imprecise here; in fact, in a bundle one also has some kind of space structure on the union of the E_p.

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